Governing equations in vector and tensor notations

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Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0$$

Or:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0$$

Momentum equation (Navier-Stokes equation):

$$\frac{\partial (\rho \boldsymbol{V})}{\partial t} + \nabla \cdot (\rho \boldsymbol{V} \otimes \boldsymbol{V}) = \rho \mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\tau}$$

Or:

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} = \rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

Energy equation (for internal energy, no kinetic energy):

$$\frac{\partial (\rho e)}{\partial t} + \nabla \cdot (\rho \boldsymbol{V} e) = -p(\nabla \cdot \boldsymbol{V}) + \tau : \nabla \boldsymbol{V} - \nabla \cdot \boldsymbol{q}$$

Or:

$$\frac{\partial (\rho e)}{\partial t} + \frac{\partial (\rho u_j e)}{\partial x_j} = -p \frac{\partial u_j}{\partial x_j} + \tau_{ij} \frac{\partial u_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j}$$

$$\frac{\partial (\rho e)}{\partial t} + \frac{\partial (\rho u_j e)}{\partial x_j} = -p \frac{\partial u_j}{\partial x_j} + \tau_{ij} \frac{\partial u_i}{\partial x_j} + \frac{1}{\partial x_j}\left(\lambda \frac{\partial T}{\partial x_j}\right) + \frac{1}{\partial x_j}\left(\sum_m \rho h_m D_m \frac{\partial Y_m}{\partial x_j}\right)$$

Species equation:

$$\frac{\partial (\rho Y_m)}{\partial t} + \nabla \cdot (\rho \boldsymbol{V} Y_m) = -\nabla \cdot (\rho Y_m V_m) + S_m$$

$$\frac{\partial (\rho Y_m)}{\partial t} + \nabla \cdot (\rho \boldsymbol{V} Y_m) = \nabla \cdot (\rho D_m \nabla Y_m) + S_m$$

$$\frac{\partial \rho Y_m}{\partial t} + \frac{\partial (\rho u_j Y_m)}{\partial x_j} = -\frac{1}{\partial x_j}(\rho Y_m V_m) + S_m$$

$$\frac{\partial \rho Y_m}{\partial t} + \frac{\partial (\rho u_j Y_m)}{\partial x_j} = \frac{1}{\partial x_j}\left(\rho D_m \frac{\partial Y_m}{\partial x_j}\right) + S_m$$

Stress tensor:

$$\boldsymbol{\tau} = \mu[\nabla \boldsymbol{V} + (\nabla \boldsymbol{V})^T] - \frac{2}{3}\mu(\nabla \cdot \boldsymbol{V})\mathbf{I}$$

Or:

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu(\frac{\partial u_k}{\partial x_k}\delta_{ij})$$

Het flux:

$$\boldsymbol{q} = -\lambda \nabla T + \sum_m \rho h_m Y_m V_m \approx -\lambda \nabla T - \sum_m \rho h_m D_m \nabla Y_m$$

Or:

$$q_j = -\lambda \frac{\partial T}{\partial x_j} + \sum_m \rho h_m Y_m V_m \approx -\lambda \frac{\partial T}{\partial x_j} - \sum_m \rho h_m D_m \frac{\partial Y_m}{\partial x_j}$$

Operation symbol (:) comes from:

$$\nabla \cdot (\boldsymbol{V} \cdot \boldsymbol{\tau}) = \boldsymbol{V} \cdot (\nabla \cdot \boldsymbol{\tau}) + \boldsymbol{\tau}\colon \nabla \boldsymbol{V}$$

First term on the right hand is cancelled when subtracting the momentum equation times density from the energy equation of total energy $(e + \frac{1}{2}V^2)$ to derive the energy equation of internal energy $e = h - p/\rho$.