Governing Equations for Fully Compressible Flow

Yang Gao

11/11/2015

***Continuity equation*:**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0$$

i.e. in scalar form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

***Momentum equation (Navier-Stokes equation)*:**

Conservation form

$$\frac{\partial (\rho \boldsymbol{V})}{\partial t} + \nabla \cdot (\rho \boldsymbol{V} \otimes \boldsymbol{V}) = \rho \mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\tau}$$

where $\boldsymbol{\tau} = \mu[\nabla \boldsymbol{V} + (\nabla \boldsymbol{V})^T] - \frac{2}{3}\mu(\nabla \cdot \boldsymbol{V})\mathbf{I}$,

$$\boldsymbol{\tau} = \mu \begin{bmatrix} 2\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla \cdot \boldsymbol{V}) & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} - \frac{2}{3}(\nabla \cdot \boldsymbol{V}) & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} - \frac{2}{3}(\nabla \cdot \boldsymbol{V}) \end{bmatrix}$$

$$\boldsymbol{\tau}_x = \mu[2\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla \cdot \boldsymbol{V}), \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}]$$

$$\boldsymbol{\tau}_y = \mu[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, 2\frac{\partial v}{\partial y} - \frac{2}{3}(\nabla \cdot \boldsymbol{V}), \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}]$$

$$\boldsymbol{\tau}_z = \mu[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, 2\frac{\partial w}{\partial z} - \frac{2}{3}(\nabla \cdot \boldsymbol{V})]$$

The $\nabla\cdot(\rho\boldsymbol{V}\otimes\boldsymbol{V})$ term:

x-component, $\nabla\cdot(\rho u\boldsymbol{V})$

y-component, $\nabla\cdot(\rho v\boldsymbol{V})$

z-component, $\nabla\cdot(\rho w\boldsymbol{V})$

The $\nabla\cdot\boldsymbol{\tau}$ term:

x-component,

$$\nabla\cdot\boldsymbol{\tau}_x=\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)\right]+\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)\right]-\frac{2}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$\nabla\cdot\boldsymbol{\tau}_x=\nabla\cdot(\mu\nabla u)+\left\{\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)\right]-\frac{2}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right\}$$

$$\boxed{\nabla\cdot\boldsymbol{\tau}_x=\nabla\cdot(\mu\nabla u)+S_{Mx}}$$

where $S_{Mx}=\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)\right]-\frac{2}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$

Assume constant viscosity μ

$$\nabla\cdot\boldsymbol{\tau}_x=\mu\nabla^2u+\mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right)-\frac{2}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]$$

$$\nabla\cdot\boldsymbol{\tau}_x=\mu\nabla^2u+\mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right)-\frac{2}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]$$

$$\nabla\cdot\boldsymbol{\tau}_x=\mu\nabla^2u+\mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)-\frac{2}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]$$

$$\nabla\cdot\boldsymbol{\tau}_x=\mu\nabla^2u+\frac{\mu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$\boxed{\nabla\cdot\boldsymbol{\tau}_x=\mu\nabla^2u+\frac{\mu}{3}\frac{\partial}{\partial x}(\nabla\cdot\boldsymbol{V})}$$

Therefore,

x-component, $\nabla\cdot\boldsymbol{\tau}_x=\nabla\cdot(\mu\nabla u)+S_{Mx}$, or constant μ, $\nabla\cdot\boldsymbol{\tau}_x=\mu\nabla^2u+\frac{\mu}{3}\frac{\partial}{\partial x}(\nabla\cdot\boldsymbol{V})$

y-component, $\nabla\cdot\boldsymbol{\tau}_y=\nabla\cdot(\mu\nabla v)+S_{My}$, or constant μ, $\nabla\cdot\boldsymbol{\tau}_y=\mu\nabla^2v+\frac{\mu}{3}\frac{\partial}{\partial y}(\nabla\cdot\boldsymbol{V})$

z-component, $\nabla\cdot\boldsymbol{\tau}_z=\nabla\cdot(\mu\nabla w)+S_{Mz}$, or constant μ, $\nabla\cdot\boldsymbol{\tau}_z=\mu\nabla^2w+\frac{\mu}{3}\frac{\partial}{\partial z}(\nabla\cdot\boldsymbol{V})$

Momentum equation in scalar form:

x-component,

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \boldsymbol{V}) = \rho \mathrm{g}_x - \frac{\partial p}{\partial x} + \nabla \cdot (\mu \nabla u) + S_{Mx}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} + \frac{\partial(\rho u v)}{\partial y} + \frac{\partial(\rho u w)}{\partial z} = \rho \mathrm{g}_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) + S_{Mx}$$

y-component,

$$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \boldsymbol{V}) = \rho \mathrm{g}_y - \frac{\partial p}{\partial y} + \nabla \cdot (\mu \nabla v) + S_{My}$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v u)}{\partial x} + \frac{\partial(\rho v v)}{\partial y} + \frac{\partial(\rho v w)}{\partial z} = \rho \mathrm{g}_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right) + S_{My}$$

z-component,

$$\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \boldsymbol{V}) = \rho \mathrm{g}_z - \frac{\partial p}{\partial z} + \nabla \cdot (\mu \nabla w) + S_{Mz}$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho w u)}{\partial x} + \frac{\partial(\rho w v)}{\partial y} + \frac{\partial(\rho w w)}{\partial z} = \rho \mathrm{g}_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial w}{\partial z}\right) + S_{Mz}$$

Momentum equation in scalar form (constant viscosity μ):

x-component,

$$\frac{\partial(\rho u)}{\partial t} + \nabla\cdot(\rho u \boldsymbol{V}) = \rho g_x - \frac{\partial p}{\partial x} + \mu\nabla^2 u + \frac{\mu}{3}\frac{\partial}{\partial x}(\nabla\cdot\boldsymbol{V})$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\mu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

y-component,

$$\frac{\partial(\rho v)}{\partial t} + \nabla\cdot(\rho v \boldsymbol{V}) = \rho g_y - \frac{\partial p}{\partial y} + \mu\nabla^2 v + \frac{\mu}{3}\frac{\partial}{\partial y}(\nabla\cdot\boldsymbol{V})$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho vu)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \frac{\mu}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

z-component,

$$\frac{\partial(\rho w)}{\partial t} + \nabla\cdot(\rho w \boldsymbol{V}) = \rho g_z - \frac{\partial p}{\partial z} + \mu\nabla^2 w + \frac{\mu}{3}\frac{\partial}{\partial z}(\nabla\cdot\boldsymbol{V})$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho wu)}{\partial x} + \frac{\partial(\rho wv)}{\partial y} + \frac{\partial(\rho ww)}{\partial z} = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \frac{\mu}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

Species equation:

$$\frac{\partial(\rho Y_k)}{\partial t} + \nabla\cdot(\rho \boldsymbol{V} Y_k) = -\nabla\cdot(\rho Y_k \boldsymbol{V}_k) + \dot{\omega}_k$$

In scalar form,

$$\frac{\partial(\rho Y_k)}{\partial t} + \left[\frac{\partial(\rho u Y_k)}{\partial x} + \frac{\partial(\rho v Y_k)}{\partial y} + \frac{\partial(\rho w Y_k)}{\partial z}\right] = -\left[\frac{\partial(\rho Y_k V_{k,x})}{\partial x} + \frac{\partial(\rho Y_k V_{k,y})}{\partial y} + \frac{\partial(\rho Y_k V_{k,z})}{\partial z}\right] + \dot{\omega}_k$$

Energy equation:

$$\rho c_p \frac{DT}{Dt} = -\sum_{k=1}^{KK} \dot{\omega}_k h_k - \nabla\cdot\boldsymbol{q} + \frac{Dp}{Dt} + \boldsymbol{\tau}:\nabla\boldsymbol{V} + body\ force\ work$$

$$\boldsymbol{q} = -\lambda\nabla T + \sum_{k=1}^{KK}(h_k Y_k \boldsymbol{V}_k) + Dufour\ effects + radiation$$

By neglecting Dufour effects, radiation and body force work,

$$\rho c_p \frac{DT}{Dt} = -\sum_{k=1}^{KK} \dot{\omega}_k h_k + \nabla\cdot(\lambda\nabla T) - \rho\sum_{k=1}^{KK}\left(c_{p,k} Y_k \boldsymbol{V}_k\right)\cdot\nabla T + \frac{Dp}{Dt} + \boldsymbol{\tau}:\nabla\boldsymbol{V}$$

Further neglect viscous dissipation $\boldsymbol{\tau}:\nabla\boldsymbol{V}$ and $\frac{Dp}{Dt}$

$$c_p\left[\frac{\partial(\rho T)}{\partial t} + \nabla\cdot(\rho\boldsymbol{V}T)\right] = -\sum_{k=1}^{KK} \dot{\omega}_k h_k + \nabla\cdot(\lambda\nabla T) - \rho\sum_{k=1}^{KK}\left(c_{p,k} Y_k \boldsymbol{V}_k\right)\cdot\nabla T$$

In scalar form,

$$c_p\left[\frac{\partial(\rho T)}{\partial t} + \frac{\partial(\rho u T)}{\partial x} + \frac{\partial(\rho v T)}{\partial y} + \frac{\partial(\rho w T)}{\partial z}\right] = -\sum_{k=1}^{KK} \dot{\omega}_k h_k + \left[\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)\right] - \left\{\rho\sum_{k=1}^{KK}\left(c_{p,k} Y_k V_{k,x}\right)\frac{\partial T}{\partial x} + \rho\sum_{k=1}^{KK}\left(c_{p,k} Y_k V_{k,y}\right)\frac{\partial T}{\partial y} + \rho\sum_{k=1}^{KK}\left(c_{p,k} Y_k V_{k,z}\right)\frac{\partial T}{\partial z}\right\}$$

If assuming all the species have the same specific heat capacity $c_{p,k}$, then the mass diffusion term is zero.

Some details for diffusion velocity $\boldsymbol{V}_k$:

$\boldsymbol{V}_k$ is the diffusion velocity of species k in the *x*-*y*-*z*-direction, by neglecting Soret effect, body-force, pressure gradient induced barodiffusion,

$$\boldsymbol{V}_k = \boldsymbol{V}'_k + \boldsymbol{V}_c = -\frac{D_k}{X_k}\nabla X_k + \boldsymbol{V}_c$$

$$\boldsymbol{V}_c = \frac{1}{W}\sum_{k=1}^{KK} D_k\, W_k \nabla X_k \; or \; \boldsymbol{V}_c = -\sum_{k=1}^{KK} Y_k\, \boldsymbol{V}'_k$$

If the last species is solved by $Y_{KK} = 1 - \sum_{k=1}^{KK-1} Y_k$, since $\sum Y_k V_k = 0$. The correction velocity $\boldsymbol{V}_c$ is not used, i.e. $\boldsymbol{V}_k = \boldsymbol{V}'_k$. The last species absorbs all the inconsistencies, typically N_2.

For mixture-averaged model, with Curtiss-Hirschfelder approximation,

$$D_k = D_k^{mix} = \frac{1 - Y_k}{\sum_{j\neq k}^{KK} X_j / D_{kj}}$$

where D_{kj} is the binary diffusion coefficient between species k and j.

In mass fraction form,

$$X_k = Y_k W / W_k,\, W = \sum X_k W_k = \left(\sum \frac{Y_j}{W_j}\right)^{-1},\, \nabla W = \sum_{j=1}^{KK} W_j \nabla X_j = -W^2 \sum_{j=1}^{KK} \frac{\nabla Y_j}{W_j}$$

Then, we transform $\boldsymbol{V}_k$ to the mass fraction form,

$$\nabla X_k = \frac{W}{W_k}\nabla Y_k + \frac{Y_k}{W_k}\nabla W = \left(\nabla Y_k + \frac{Y_k}{W}\nabla W\right)\frac{W}{W_k} = \left(\nabla Y_k + \frac{Y_k}{W}\nabla W\right)\frac{X_k}{Y_k}$$

$$-\frac{D_{km}}{X_k}\nabla X_k = -\frac{D_k}{Y_k}\left(\nabla Y_k + \frac{Y_k}{W}\nabla W\right) = -\frac{D_k}{Y_k}\nabla Y_k - \frac{D_k}{W}\nabla W$$

$$\begin{aligned}\boldsymbol{V}_k = -\frac{D_k}{X_k}\nabla X_k + \boldsymbol{V}_c &= \left(-\frac{D_k}{Y_k}\nabla Y_k - \frac{D_k}{W}\nabla W\right) + \frac{1}{W}\sum_{j=1}^{KK} D_k\, W_k \nabla X_k \\ &= \left(-\frac{D_k}{Y_k}\nabla Y_k - \frac{D_k}{W}\nabla W\right) + \frac{1}{W}\sum_{j=1}^{KK} D_k\, W_k \left(\nabla Y_k + \frac{Y_k}{W}\nabla W\right)\frac{W}{W_k} \\ &= \left(-\frac{D_k}{Y_k}\nabla Y_k - \frac{D_k}{W}\nabla W\right) + \left(\sum_{j=1}^{KK} D_k\, \nabla Y_k + \frac{\nabla W}{W}\sum_{j=1}^{KK} D_k\, Y_k\right) \\ &= \left(-\frac{D_k}{Y_k}\nabla Y_k\right) + \left(-\frac{D_k}{W}\nabla W + \sum_{j=1}^{KK} D_k\, \nabla Y_k + \frac{\nabla W}{W}\sum_{j=1}^{KK} D_k\, Y_k\right) = \left(-\frac{D_k}{Y_k}\nabla Y_k\right) + \boldsymbol{V}_c{}'\end{aligned}$$

Then $Y_k \boldsymbol{V}_k = -D_k \nabla Y_k + Y_k \boldsymbol{V}_c{}'$ in the mass fraction form.

Some details in stress tensor $\boldsymbol{\tau}$:

For Newtonian fluid,

$$\boldsymbol{\tau} = 2\mu\boldsymbol{\sigma} + \lambda(\nabla \cdot \boldsymbol{V})\mathbf{I} = \mu\left\{[\nabla\boldsymbol{V} + (\nabla\boldsymbol{V})^T] - \frac{2}{3}(\nabla \cdot \boldsymbol{V})\mathbf{I}\right\} + \lambda(\nabla \cdot \boldsymbol{V})\mathbf{I}$$

$$\boldsymbol{\sigma} = \frac{1}{2}[\nabla\boldsymbol{V} + (\nabla\boldsymbol{V})^T] - \frac{1}{3}(\nabla \cdot \boldsymbol{V})\mathbf{I}$$

μ is dynamic viscosity.

λ is secondary viscosity/bulk viscosity/dilatational viscosity.

The first term $2\mu\boldsymbol{\sigma}$ is deviatoric stress, related to flow deformation, and $\boldsymbol{\sigma}$ is strain rate.

The second term $\lambda(\nabla \cdot \boldsymbol{V})\mathbf{I}$ is isotropic stress, related to flow expansion/dilatational.

Note that rotation does not induce viscous force.

The stress can also be written as

$$\boldsymbol{\tau} = \mu[\nabla\boldsymbol{V} + (\nabla\boldsymbol{V})^T] + \left(\lambda - \frac{2}{3}\mu\right)(\nabla \cdot \boldsymbol{V})\mathbf{I}$$

The bulk viscosity, $\lambda(\nabla \cdot \boldsymbol{V})\mathbf{I}$, is often neglected, i.e. λ is set to be 0.

In incompressible ($\nabla \cdot \boldsymbol{V} = \mathbf{0}$), neglecting the bulk viscosity

$$\boldsymbol{\tau} = 2\mu\boldsymbol{\sigma} = 2\mu\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

In incompressible ($\nabla \cdot \boldsymbol{V} = \mathbf{0}$) AND irrotational flow ($\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\Omega} = 0$, $\boldsymbol{\Omega}$ is vorticity), the bulk viscosity can be neglected, resulting in

$$\boldsymbol{\tau} = 2\mu\nabla\boldsymbol{V} = 2\mu\begin{bmatrix} \frac{\partial u}{\partial x} & 0 & 0 \\ 0 & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & \frac{\partial w}{\partial z} \end{bmatrix}$$